SEPARATION AXIOMS IN INFRA SOFT BITOPOLOGICAL SPACES

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ABSTRACT. The purpose of this paper is to introduce and study some separation axioms in infra soft bitopological spaces and investigate some of their important characterizations.

1. INTRODUCTION

The concept of soft sets was first introduced by Molodtsov [24] in 1999 as a general mathematical tool for dealing with uncertain objects while modelling the problems with incomplete information in engineering, physics, computer science, economics, social sciences and medical sciences. In [25], Molodtsov applied soft sets successfully in directions such as smoothness of functions, game theory, operations research, Riemann-integration, Perron integration, probability and theory of measurement. Maji et. al [22] applied soft sets in a multicriteria decision making problems. It is based on the notion of knowledge reduction of rough sets. They applied the technique of knowledge reduction to the information table induced by the soft set. In [23], they defined and studied several basic notions of soft set theory. In 2005, Pei and Miao [29] and Chen [5] improved the work of Maji et.al [22-23]. A. Kharal and B. Ahmad [20] defined and discussed the several properties of soft images and soft inverse images of soft sets. They also applied these notions to the problem of medical diagnosis in medical systems. Many researchers have contributed towards the algebraic structure of soft set theory [1-2],[4], [7], [10-19], [21], [28], [31-32]. In 2011, Shabir and Naz [31] initiated the study of soft topological spaces. Also in 2011, S. Hussain and B. Ahmad [9] continued investigating the properties of soft open(closed), soft neighborhood and soft closure. They also defined and discussed the properties of soft interior, soft exterior and soft boundary. Shabir et. al [31] and D. N. Georgiou et. al [7], defined and studied some soft separation axioms, soft -continuity and soft connectedness in soft spaces using (ordinary) points of a topological space X. Resently, Jamal M. Mustafa [26, 27], defined and studied supra soft b-compact, supra soft b-Lindelof, supra soft b-R_0 and supra soft b-R_1 spaces . As a generalized of soft topological spaces, T. M. Al-shami [3] introduced the notion of infra soft topological spaces. After the concept of bitopological spaces was introduced by J.C. Kelly [20] as an extension of topological spaces in 1963, B.M. Ittanagi [10] defined the notion of soft bitopological space. A.F. Sayed [31] studied separation axioms in fuzzy soft bitopological spaces. A study of soft bitopological spaces is a generalization of the study of soft topological spaces as every soft bitopological space (X, μ_1, μ_2, E) can be regarded as a soft topological space (X, μ, E) if $\mu_1 = \mu_2 = \mu$. In this paper we define separation axioms in infra soft bitopological spaces and investigate some of their important characterizations.

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2. PRELIMINARIES

In this section we will introduce necessary definitions and theorems for soft sets.

Definition 2.1. [24] Let X be an initial universe and E a set of parameters. Let $P(X)$ denote the power set of X. A pair (F, E) is called a soft set over X, where F is a mapping given by $F : E \to P(X)$. In other words, a soft set over X is a parameterized family of subsets of the universe X. For $e \in E$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, E).

Definition 2.2. [24] The complement of a soft set (F, E) is denoted by $(F, E)^c$ and is defined by $(F, E)^c = (F^c, E)$ where, $F^c : E \to P(X)$ is a mapping given by $F^c(e) = X - F(e)$, for all $e \in E$.

Let us call F^c to be the soft complement function of F. Clearly $(F^c)^c$ is the same as F and $((F, E)^c)^c = (F, E)$.

Definition 2.3. [35] A soft set (F, E) over X is said to be a null soft set, denoted by $\widetilde{\phi}$, if for all $e \in E$, $F(e) = \phi$. Clearly, $(\widetilde{\phi}^c)^c = \phi$.

Definition 2.4. [35] A soft set (F, E) over X is said to be an absolute soft set, denoted by $\widetilde{X}$, if for all $e \in E$, $F(e) = X$. Clearly, $\widetilde{X}^c = \widetilde{\phi}$.

Definition 2.5. [23] Let (F, E), (G, E) be two soft sets over X. Then,

(1) (F, E) is a soft subset of (G, E), denoted by $(F, E)\widetilde{\subseteq}(G, E)$, if $F(e) \subseteq G(e)$, $\forall e \in E$. In this case, (F, E) is said to be a soft subset of (G, E) and (G, E) is said to be a soft superset of (F, E).

(2) Two soft sets (F, E) and (G, E) over a common universe set X are said to be equal, denoted by $(F, E) = (G, E)$, if $F(e) = G(e)$, $\forall e \in E$.

(3) The union of two soft sets (F, E) and (G, E) over the common universe X, denoted by $(F, E)\widetilde{\cup}(G, E)$, is the soft set (H, E), where$H(e) = F(e) \cup G(e)$, for all $e \in E$.

(4) The intersection of two soft sets (F, E) and (G, E) over the common universe X, denoted by $(F, E)\widetilde{\cap}(G, E)$, is the soft set (M, E), where $M(e) = F(e) \cap G(e)$, for all $e \in E$.

Definition 2.6. [31] Let (F, E) be a soft set over X and $x \in X$. We say that $x \in (F, E)$ read as x belongs to the soft set (F, E) whenever $x \in F(e)$ for all $e \in E$.

Note that for any $x \in X$, $x \notin (F, E)$ if $x \notin F(e)$ for some $e \in E$.

Definition 2.7. [31] Let τ be a collection of soft sets over a universe X with a fixed set of parameters E, then τ is called a soft topology on X if

(1) $\widetilde{\phi}$, $\widetilde{X} \in \tau$.

(2) the union of any number of soft sets in τ belongs to τ.

(3) the intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a soft topological space over X.

Definition 2.8. [31] Let (X, τ, E) be a soft topological space, and (F, E) be a soft set over X. If Y is a non empty subset of X, then the sub soft set of (F, E) over Y denoted by (F_Y, E), is defined as follows:

$F_Y(e) = Y \cap F(e)$ for all $e \in E$.

Definition 2.9. [31] Let (X, τ, E) be a soft topological space and Y be a non empty subset of X. Then, $\tau_Y = \{(F_Y, E) : (F, E) \in \tau\}$ is called the soft relative topology on Y and (Y, τ_Y, E) is called a soft subspace of (X, τ, E).

Definition 2.10. [3] Let μ be a collection of soft sets over a universe X with a fixed set of parameters E. Then μ is called an infra soft topology on X with a fixed set E if:

(1) $\widetilde{\phi} \in \mu$

(2) The finite intersection of soft sets in μ belongs to μ.

(X, μ, E) is called infra soft topological space and the elements of μ are called infra soft open sets. The soft complement of any infra soft open set is called infra soft closed.

Definition 2.11. Let (X, μ, E) be an infra soft topological space and Y be a non empty subset of X. Then, $\mu_Y = \{(F_Y, E) : (F, E) \in \mu\}$ is called the infra soft relative topology on Y and (Y, μ_Y, E) is called an infra soft subspace of (X, μ, E).

Definition 2.12. [3] Let (X, μ, E) be an infra soft topological space and (F, E) be a soft set over X. Then the infra soft interior of (F, E) , denoted by $int^i(F, E)$ is the soft union of all infra soft open subsets of (F, E) .

Definition 2.13. [3] Let (X, μ, E) be an infra soft topological space and (F, E) be a soft set over X. Then the infra soft closure of (F, E) , denoted by $cl^i(F, E)$ is the soft intersection of all infra soft closed sets containing (F, E) .

3. Separation Axioms in infra Soft Bitopological Spaces

Let X be a nonempty set E be a set of parameters and μ_1, μ_2 be two infra soft topologies on X. Then (X, μ_1, μ_2, E) is said to be an infra soft bitopological space (briefly ISBTS). Let (X, μ_1, μ_2, E) be an infra soft bitopological space and (F, E) be a soft set. The infra soft closure of (F, E) with respect to μ_n are denoted by $cl^i_{\mu_n}(F, E)$, for $n = 1, 2$.

Definition 3.1. Let (X, μ_1, μ_2, E) be an ISBTS over X. (X, μ_1, μ_2, E) is said to be an infra soft $\mu_{m,n} - T_0$ space if for any points x, y with $x \neq y$ there exist infra soft open sets (F, E), $(G, E) \in \mu_1$ and (H, E), $(K, E) \in \mu_2$ such that

$x \in (F, E)$, $y \notin (F, E)$ and $x \in (H, E)$, $y \notin (H, E)$ or

$y \in (G, E)$, $x \notin (G, E)$ and $y \in (K, E)$, $x \notin (K, E)$ for $m, n = 1, 2$.

Definition 3.2. Let (X, μ_1, μ_2, E) be an ISBTS over X. (X, μ_1, μ_2, E) is said to be an infra soft $\mu_{m,n} - T_1$ space if for any points x, y with $x \neq y$ there exist infra soft open sets $(F, E) \in \mu_m$ and $(G, E) \in \mu_n$ such that

$x \in (F, E)$, $y \notin (F, E)$ and $y \in (G, E)$, $x \notin (G, E)$ for $m, n = 1, 2$.

Remark 3.3. Every infra soft $\mu_{m,n} - T_1$ space is infra soft $\mu_{m,n} - T_0$, but the converse need not be true as shown in the following exampll.

Example 3.4. *Let* $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ *and* $\mu_1 = \{\widetilde{\phi}, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F$ *and* $\mu_2 = \{\widetilde{\phi}, \widetilde{X}, (G_1, E), (G_2, E), (G_3, E), (G_4, E), (G_5, E), (G_6, E)\}$, *where* (F_1, E), (F_2, E),

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(F_3, E), (F_4, E), (F_5, E), (G_1, E), (G_2, E), (G_3, E), (G_4, E), (G_5, E) and (G_6, E) are soft sets over X, defined as follows: $F_1(e_1) = \{h_2\}$, $F_1(e_2) = \{h_1\}$, $F_2(e_1) = \{h_2, h_3\}$, $F_2(e_2) = \{h_1, h_2\}$, $F_3(e_1) = \{h_1, h_2\}$, $F_3(e_2) = X$, $F_4(e_1) = \{h_1, h_2\}$, $F_4(e_2) = \{h_1, h_3\}$, $F_5(e_1) = \{h_2\}$, $F_5(e_2) = \{h_1, h_2\}$ and $G_1(e_1) = \{h_2\}$, $G_1(e_2) = \{h_2\}$, $G_2(e_1) = \{h_3\}$, $G_2(e_2) = \{h_3\}$, $G_3(e_1) = \{h_2, h_3\}$, $G_3(e_2) = \{h_2, h_3\}$, $G_4(e_1) = \{h_1, h_2\}$, $G_4(e_2) = \{h_1, h_2\}$, $G_5(e_1) = \{h_1, h_3\}$, $G_5(e_2) = \{h_1, h_3\}$, $G_6(e_1) = \{h_1\}$, $G_6(e_2) = \{h_1\}$. Then μ_1 and μ_2 are infra soft topologies on X. Therefore (X, μ_1, μ_2, E) is an ISBTS over X. One can easily see that (X, μ_1, μ_2, E) is an infra soft $\mu_{m,n} - T_0$ space but not infra soft $\mu_{m,n} - T_1$.

Theorem 3.5. *Let (X, μ_1, μ_2, E) be an ISBTS over X. Then (X, μ_1, μ_2, E) is an infra soft $\mu_{m,n} - T_1$ space if and only if the set $\{x\}$ is a μ_m infra soft closed set and a μ_n infra soft closed set, for all $x \in (X, \mu_1, \mu_2, E)$.*

Proof. Assume that (X, μ_1, μ_2, E) is an infra soft $\mu_{m,n} - T_1$ space and $x \in (X, \mu_1, \mu_2, E)$. For each $y \in \{x\}^c$, $x \neq y$. By assumption, there exist infra soft open sets $(F, E) \in \mu_m$ and $(G, E) \in \mu_n$ such that $x \in (F, E)$, $y \notin (F, E)$ and $y \in (G, E)$, $x \notin (G, E)$ for $m, n = 1, 2$. Then the soft set $\{x\}^c$ is a μ_m infra soft open set and a μ_n infra soft open set. So, $\{x\}$ is a μ_m infra soft closed set and a μ_n infra soft closed set.

Conversely, assume that $\{x\}$ is a μ_m infra soft closed set and a μ_n infra soft closed set, for all $x \in (X, \mu_1, \mu_2, E)$. Let $x, y \in (X, \mu_1, \mu_2, E)$ with $x \neq y$. By assumption, we obtain that $\{x\}$ is a μ_n infra soft closed set and $\{y\}$ is a μ_m infra soft closed set. Let $(F, E) = \widetilde{X} - \{y\}$ and $(G, E) = \widetilde{X} - \{x\}$. Then (F, E) is a μ_m infra soft open set and (G, E) is a μ_n infra soft open set. Therefore $x \in (F, E)$, $y \notin (F, E)$ and $y \in (G, E)$, $x \notin (G, E)$ are obtained. Then (X, μ_1, μ_2, E) is an infra soft $\mu_{m,n} - T_1$ space. □

Definition 3.6. Let (X, μ_1, μ_2, E) be an ISBTS over X. (X, μ_1, μ_2, E) is said to be an infra soft $\mu_{m,n} - T_2$ space if for any points x, y with $x \neq y$ there exist infra soft open sets $(F, E) \in \mu_m$ and $(G, E) \in \mu_n$ such that

$x \in (F, E)$, $y \in (G, E)$ and $(F, E)\widetilde{\cap}(G, E) = \phi$, for $m, n = 1, 2$.

Remark 3.7. Every infra soft $\mu_{m,n} - T_2$ space is infra soft $\mu_{m,n} - T_1$, but the converse need not be true as shown in the following example.

Example 3.8. *Let $X = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2\}$ and $\mu_1 = \{\widetilde{\phi}, \widetilde{X}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F$ and $\mu_2 = \{\widetilde{\phi}, \widetilde{X}, (G_1, E), (G_2, E), (G_3, E), (G_4, E), (G_5, E), (G_6, E)\}$, where (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (F_6, E), (F_7, E), (G_1, E), (G_2, E), (G_3, E), (G_4, E), (G_5, E) and (G_6, E) are soft sets over X, defined as follows: $F_1(e_1) = \{h_1\}$, $F_1(e_2) = \{h_1, h_3\}$, $F_2(e_1) = \{h_3\}$, $F_2(e_2) = \{h_3\}$, $F_3(e_1) = \{h_1, h_3\}$, $F_3(e_2) = \{h_1, h_3\}$, $F_4(e_1) = \{\}$, $F_4(e_2) = \{h_3\}$, $F_5(e_1) = \{h_2, h_3\}$, $F_5(e_2) = \{h_2\}$, $F_6(e_1) = \{h_3\}$, $F_6(e_2) = \{\}$, $F_7(e_1) = \{h_2, h_3\}$, $F_7(e_2) = \{h_2, h_3\}$, and $G_1(e_1) = \{h_2\}$, $G_1(e_2) = \{h_2\}$, $G_2(e_1) = \{h_3\}$, $G_2(e_2) = \{h_3\}$, $G_3(e_1) = \{h_2, h_3\}$, $G_3(e_2) = \{h_2, h_3\}$, $G_4(e_1) = \{h_1, h_2\}$, $G_4(e_2) = \{h_1, h_2\}$, $G_5(e_1) = \{h_1, h_3\}$, $G_5(e_2) = \{h_1, h_3\}$, $G_6(e_1) = \{h_1\}$, $G_6(e_2) = \{h_1\}$. Then μ_1 and μ_2 are infra soft topologies on X. Therefore (X, μ_1, μ_2, E) is an ISBTS over X. One can easily see that (X, μ_1, μ_2, E) is an infra soft $\mu_{m,n} - T_1$ space but not infra soft $\mu_{m,n} - T_2$.*

Definition 3.9. Let (X, μ_1, μ_2, E) be an ISBTS over X. Then (X, μ_1, μ_2, E) is said to be an infra soft $\mu_{m,n}$−regular space if for any point x and for any μ_m infra soft closed set

(F,E) with $x \notin (F,E)$, there exist $(G,E) \in \mu_m$ and $(H,E) \in \mu_n$ such that $x \in (G,E)$, $(F,E)\widetilde{\subseteq}(H,E)$ and $(G,E)\widetilde{\cap}(H,E) = \phi$. Also (X,μ_1,μ_2,E) is said to be an infra soft $\mu_{m,n} - T_3$ space if (X,μ_1,μ_2,E) is infra soft $\mu_{m,n} - T_1$ and infra soft $\mu_{m,n}-$regular.

Theorem 3.10. *Let (X,μ_1,μ_2,E) be an ISBTS over X. Then the following are equivalent:*

1) (X,μ_1,μ_2,E) is an infra soft $\mu_{m,n}-$regular space.

2) If $x \in (X,\mu_1,\mu_2,E)$ and (F,E) is a μ_m infra soft closed set with $x \notin (F,E)$, then there exist $(G,E) \in \mu_m$ and $(H,E) \in \mu_n$ such that $x \in (G,E)$, $(F,E)\widetilde{\subseteq}(H,E)$ and $cl^i_{\mu_n}(G,E)\widetilde{\cap}(H,E) = \phi$.

3) If $x \in (X,\mu_1,\mu_2,E)$ and (F,E) is a μ_m infra soft closed set with $x \notin (F,E)$, then there is a μ_m infra soft open set (G,E) containing x such that $cl^i_{\mu_n}(G,E)\widetilde{\cap}(F,E) = \phi$.

4) If $x \in (X,\mu_1,\mu_2,E)$ and $(G,E) \in \mu_m$ with $x \in (G,E)$, then there is a μ_m infra soft open set (H,E) containing x such that $x \in (H,E)\widetilde{\subseteq}cl^i_{\mu_n}(H,E)\widetilde{\subseteq}(G,E)$.

Proof. 1) $\implies$ 2) Let $x \in (X,\mu_1,\mu_2,E)$ and (F,E) be a μ_m infra soft closed set with $x \notin (F,E)$. Since (X,μ_1,μ_2,E) is infra soft $\mu_{m,n}-$regular, there exist $(G,E) \in \mu_m$ and $(H,E) \in \mu_n$ such that $x \in (G,E)$, $(F,E)\widetilde{\subseteq}(H,E)$ and $(G,E)\widetilde{\cap}(H,E) = \phi$. Suppose that $cl^i_{\mu_n}(G,E)\widetilde{\cap}(H,E) \neq \phi$. Then the point $y \in cl^i_{\mu_n}(G,E)\widetilde{\cap}(H,E)$ is a soft tangency point of (G,E) in μ_n- infra soft topology. Since y∈(H,E)∈ μ_n, $(G,E)\widetilde{\cap}(H,E) \neq \phi$, is obtained. This is a contradiction.

2) $\implies$ 3) It is clear.

3) $\implies$ 4). Assume that $x \in (X,\mu_1,\mu_2,E)$ and $(G,E) \in \mu_m$ with $x \in (G,E)$, Then $(G,E)^c$ is a μ_m infra soft closed set and $x \notin (G,E)^c$, By 3), there exists a μ_m infra soft open set (H,E) containing x such that $cl^i_{\mu_n}(H,E)\widetilde{\cap}(G,E)^c = \phi$. Hence $x \in (H,E)\widetilde{\subseteq}cl^i_{\mu_n}(H,E)\widetilde{\subseteq}(G,E)$ is obtained.

4) $\Rightarrow$ 1) Let $x \in (X,\mu_1,\mu_2,E)$ and (F,E) be a μ_m infra soft closed set with $x \notin (F,E)$. Then $x \in (F,E)^c$ and $(F,E)^c \in \mu_m$. By 4) there exists $(H,E) \in \mu_n$ such that $x \in (H,E)\widetilde{\subseteq}cl^i_{\mu_n}(H,E)\widetilde{\subseteq}(F,E)^c$, Moreover, $x \in (H,E)$, $(F,E)\widetilde{\subseteq}cl^i_{\mu_n}(H,E)^c \in \mu_n$ and $(H,E)\widetilde{\cap}cl^i_{\mu_n}(H,E)^c = \phi$ is satisfied. Thus (X,μ_1,μ_2,E) is an infra soft $\mu_{m,n}-$regular space. □

Definition 3.11. Let (X,μ_1,μ_2,E) be an ISBTS over X. Then (X,μ_1,μ_2,E) is said to be an infra soft $\mu_{m,n}-$normal space if for any μ_m infra soft closed set (F_1,E) and for any μ_n infra soft closed set (F_2,E) with $(F_1,E)\widetilde{\cap}(F_2,E) = \phi$, there exist $(G_1,E) \in \mu_m$ and $(G_2,E) \in \mu_n$ such that $(F_1,E)\widetilde{\subseteq}(G_2,E)$, $(F_2,E)\widetilde{\subseteq}(G_1,E)$ and $(G_1,E)\widetilde{\cap}(G_2,E) = \phi$.

If (X,μ_1,μ_2,E) is an infra soft $\mu_{m,n}-$normal space and infra soft $\mu_{m,n} - T_1$ space, then (X,μ_1,μ_2,E) is said to be an infra soft $\mu_{m,n} - T_4$ space.

Theorem 3.12. *Every infra soft $\mu_{m,n} - T_4$ space is infra soft $\mu_{m,n} - T_3$.*

Proof. Let (X,μ_1,μ_2,E) be an infra soft $\mu_{m,n} - T_4$ space. Let $x \in (X,\mu_1,\mu_2,E)$ and (F,E) be a μ_m infra soft closed set such that $x \notin (F,E)$, Since (X,μ_1,μ_2,E) is an infra soft $\mu_{m,n} - T_1$ space, $\{x\}$ is a μ_n infra soft closed set. Since $\{x\}\widetilde{\cap}(F,E) = \phi$ and (X,μ_1,μ_2,E) is an infra soft $\mu_{m,n}-$normal space, there exist a μ_m infra soft open set (G,E) and μ_n infra soft open set (H,E) such that $x \in (G,E)$, $(F,E)\widetilde{\subseteq}(H,E)$ and $(G,E)\widetilde{\cap}(H,E) = \phi$. Hence (X,μ_1,μ_2,E) be an infra soft $\mu_{m,n} - T_3$ space. □

Theorem 3.13. *Let (X,μ_1,μ_2,E) be an ISBTS over X. Then the following conditions are equivalent:*

1) (X,μ_1,μ_2,E) is an infra soft $\mu_{m,n}-$normal space.

2) For any μ_m infra soft closed set (F_1,E) and for any μ_n infra soft closed set (F_2,E) with $(F_1,E)\widetilde{\cap}(F_2,E) = \phi$, there exist $(G_1,E) \in \mu_m$ and $(G_2,E) \in \mu_n$ such that $(F_1,E)\widetilde{\subseteq}(G_2,E)$, $(F_2,E)\widetilde{\subseteq}(G_1,E)$ and $cl^i_{\mu_n}(G_1,E)\widetilde{\cap}(G_2,E) = \phi$.

3) For any μ_m infra soft closed set (F_1,E) and for any μ_n infra soft closed set (F_2,E) with $(F_1,E)\widetilde{\cap}(F_2,E) = \phi$, there exist $(G,E) \in \mu_m$ such that $(F_2,E)\widetilde{\subseteq}(G,E)$ and $cl^i_{\mu_n}(G,E)\widetilde{\cap}(F_1,E) = \phi$.

4) For any μ_m infra soft closed set (F,E) and for any μ_n infra soft open set (G,E) with $(F,E)\widetilde{\subseteq}(G,E)$, there exists a μ_n infra soft open set (H,E) such that $(F,E)\widetilde{\subseteq}(H,E)\widetilde{\subseteq}cl^i_{\mu_m}(H,E)\widetilde{\subseteq}(G,E$

Proof. 1) $\Longrightarrow$ 2) Let (X,μ_1,μ_2,E) be an infra soft $\mu_{m,n}-$normal space, (F_1,E) be any μ_m infra soft closed set and (F_2,E) be any μ_n infra soft closed set with $(F_1,E)\widetilde{\cap}(F_2,E) = \phi$. Since (X,μ_1,μ_2,E) be an infra soft $\mu_{m,n}-$normal space, there exist $(G_1,E) \in \mu_m$ and $(G_2,E) \in \mu_n$ such that $(F_1,E)\widetilde{\subseteq}(G_2,E)$, $(F_2,E)\widetilde{\subseteq}(G_1,E)$ and $(G_1,E)\widetilde{\cap}(G_2,E) = \phi$ Assume that $cl^i_{\mu_n}(G_1,E)\widetilde{\cap}(G_2,E) \neq \phi$. Let $y \in cl^i_{\mu_n}(G_1,E)\widetilde{\cap}(G_2,E)$. Then $y \in cl^i_{\mu_n}(G_1,E)$ and sence $y \in (G_2,E) \in \mu_n$, we have $(G_1,E)\widetilde{\cap}(G_2,E) \neq \phi$. This is a contradiction. Hence $cl^i_{\mu_n}(G_1,E)\widetilde{\cap}(G_2,E) = \phi$ is obtained.

2) $\Longrightarrow$ 3) It is clear.

3) $\Longrightarrow$ 4) Assume that (F,E) is a μ_m infra soft closed set and (G,E) is a μ_n infra soft open set and $(F,E)\widetilde{\subseteq}(G,E)$. Then $(G,E)^c$ is a μ_n infra soft closed set and $(F,E)\widetilde{\cap}(G,E)^c = \phi$. By 3), there exists a μ_m infra soft open set (H,E) such that $(G,E)^c\widetilde{\subseteq}(H,E)$ and $cl^i_{\mu_n}(H,E)\widetilde{\cap}(F,E) = \phi$. Hence $(F,E)\widetilde{\subseteq}cl^i_{\mu_n}(H,E)^c\widetilde{\subseteq}(H,E)^c\widetilde{\subseteq}(G,E)$. Let $(P,E) = (cl^i_{\mu_n}(H,E))^c$. Hence (P,E) is a μ_n infra soft open set and $(F,E)\widetilde{\subseteq}(P,E)\widetilde{\subseteq}cl^i_{\mu_m}(P,E)\widetilde{\subseteq}(G$

4) $\Longrightarrow$ 1) Let (F_1,E) be a μ_m infra soft closed set and (F_2,E) be a μ_n infra soft closed set such that $(F_1,E)\widetilde{\cap}(F_2,E) = \phi$. Then $(F_2,E)^c$ is a μ_n infra soft open set and $(F_1,E)\widetilde{\subseteq}(F_2,E)^c$. By 4), there exists a μ_n infra soft open set (H,E) such that $(F_1,E)\widetilde{\subseteq}(H,E)\widetilde{\subseteq}cl^i_{\mu_m}(H,E)\widetilde{\subseteq}(F_2,E)^c$. Then $(F_1,E)\widetilde{\subseteq}(H,E) \in \mu_n$, $(F_2,E)\widetilde{\subseteq}(cl^i_{\mu_m}(H,E))^c \in \mu_m$ and $(H,E)\widetilde{\cap}(cl^i_{\mu_m}(H,E))^c = \phi$. Hence (X,μ_1,μ_2,E) is an infra soft $\mu_{m,n}-$normal space. □

Definition 3.14. Let (X,μ,E) be an infra soft topological space, and $Y \subseteq X$. Then the family $\mu_Y = \{\widetilde{Y}\widetilde{\cap}(F,E) : (F,E) \in \mu\}$ is an infra soft topology on Y. (Y,μ_Y,E) is said to be an infra soft topological subspace of (X,μ,E).

Definition 3.15. Let (X,μ_1,μ_2,E) be an ISBTS over X and $Y \subseteq X$. Then (Y,μ_{1Y},μ_{2Y},E) is called an infra soft bitopological subspace of (X,μ_1,μ_2,E).

Theorem 3.16. *Let (X,μ,E) be an infra soft topological space and (Y,μ_Y,E) be an infra soft subspace of (X,μ,E). Then $(H,E) \subseteq (Y,\mu_Y,E)$ is an infra soft closed set if and only if there exists an infra soft closed set $(F,E) \subseteq (X,\mu,E)$ such that $(H,E) = \widetilde{Y}\widetilde{\cap}(F,E)$.*

Proof. Let $(H,E) \subseteq (Y,\mu_Y,E)$ be an infra soft closed set. Then $(H,E)^c = \widetilde{Y} - (H,E)$ is an infra soft open set. Thus there exists an infra soft open set (G,E) such that $(H,E)^c = \widetilde{Y}\widetilde{\cap}(G,E)$. This implies that $(H,E) = \widetilde{Y}\widetilde{\cap}(G,E)^c$ and $(G,E)^c$ is an infra soft closed set.

Conversely, let $(H,E) = \widetilde{Y}\widetilde{\cap}(F,E)$ and (F,E) be an infra soft closed set. Then $(H,E)^c = \widetilde{Y}\widetilde{\cap}(F,E)^c$ is an infra soft open set. Thus (H,E) is an infra soft closed set in (Y,μ_Y,E). □

Theorem 3.17. *Let (X,μ_1,μ_2,E) be an ISBTS over X. If (X,μ_1,μ_2,E) is an infra soft $\mu_{m,n} - T_i$ space, then the infra soft bitopological subspace (Y,μ_{1Y},μ_{2Y},E) of (X,μ_1,μ_2,E) is an infra soft $\mu_{m,n} - T_i$ space, for $i = 0,1,2,3$.*

Proof. Let $x,\ y \in (Y,\mu_{1Y},\mu_{2Y},E)$ such that $x \neq y$. Thus there exists $(F,E) \in \mu_m$, $(G,E) \in \mu_n$ which satisfying conditions of infra soft $\mu_{m,n} - T_i$ space, for $i = 0,1,2$. Let $x \in (F,E)$ and $y \in (G,E)$. Then $x \in (F,E)\widetilde{\cap}\widetilde{Y}$, $y \in (G,E)\widetilde{\cap}\widetilde{Y}$. Also the infra soft open sets $(F,E)\widetilde{\cap}\widetilde{Y}$ and $(G,E)\widetilde{\cap}\widetilde{Y}$ in (Y,μ_{1Y},μ_{2Y},E) satisfying conditions of infra soft $\mu_{m,n} - T_i$ space, for $i = 0,1,2$.

If $i = 3$, the proof is done similarly. □

Theorem 3.18. *Let (X,μ_1,μ_2,E) be an ISBTS over X. If (X,μ_1,μ_2,E) is an infra soft $\mu_{m,n} - T_4$ space and $\widetilde{Y}$ be a μ_m infra soft closed set, μ_n infra soft closed set, then the infra soft bitopological subspace (Y,μ_{1Y},μ_{2Y},E) of (X,μ_1,μ_2,E) is an infra soft $\mu_{m,n} - T_4$ space.*

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